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ON HANDLEBODIES

V. Poénaru†

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§1. INTRODUCTION—STATEMENT OF THE MAIN RESULTS

THE SOLUTION of the generalized Poincaré Conjecture (see, for example, [3], [4]) has roughly speaking, two distinct steps, a hard one, and an easy one:

(i) From the homotopy assumptions on an n-homotopy disk Δ_n, one deduces the existence of a certain type of handlebody decomposition for Δ_n.

(ii) One shows that for that certain type of handlebody decomposition, the successive handles cancel (annihilate) each other.

The reverse of step (i) (i.e. to deduce the homotopy type from a handle-decomposition) is trivial. The present paper, deals, roughly speaking, with the reverse of step (ii). We start with a handle-body decomposition of D_n, and we show that it has to be roughly, of that type mentioned in the end of step (i).

We make things now more precise. First of all, to avoid any confusion, all manifolds we consider are differentiable, compact, orientable. All maps considered are differentiable. We start by recalling some well-known definitions. We consider M_n, a compact manifold with boundary $\partial M_n \neq \phi$ and k copies of D_n,

$$D_n{}^i = D_\lambda{}^i \times D^i_{n-\lambda} \qquad\qquad i = 1, \dots, k$$

where λ is some number $0 \leq \lambda \leq n$.

We consider

$$\partial D_n{}^i = \partial D_\lambda{}^i \times D^i_{n-\lambda} + D_\lambda{}^i \times \partial D^i_{n-\lambda} = S^i_{\lambda-1} \times D^i_{n-\lambda} + D_\lambda{}^i \times S^i_{n-\lambda-1}.$$

For each $S^i_{\lambda-1} \times D^i_{n-\lambda}$ we give a differentiable embedding:

$$f_i = S^i_{\lambda-1} \times D^i_{n-\lambda} \to \partial M_n.$$

We assume that Image$f_i \cap$ Image$f_j = \phi$ (if $i \neq j$). We consider the differentiable manifold obtained by adding to M_n the k handles of index λ, $D_\lambda \times D_{n-\lambda}$ $(i = 1, \dots, k)$, along f_i $(i = 1, \dots, k)$, and denote it by:

$$M_n + (f_1) + (f_2) + \cdots + (f_k). \tag{1}$$

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We will not distinguish between the handle-decomposition (1) and those which we can obtain from it by global isotopies of ∂M_n. We consider the following *elementary sliding operation* which can be performed on the handle-decomposition (1): We fix our attention on f_i and f_j, say $(i \neq j)$.

We consider $M_n + (f_j)$, and remark that:

$$\partial(M_n + (f_j)) = (\partial(M_n) - 1_m f_j + D_\lambda{}^j \times S^j_{n-\lambda-1}$$

$$\partial(M_n + (f_j)) \cap \partial M_n = C1(\partial M_n - \text{Image } f_j).$$

We consider an isotopy

$$\Phi : I \to \text{Emb}\,(S^i_{\lambda-1} \times D^i_{n-\lambda} \to \partial(M_n + (f_j)))$$

such that:

(i) $\Phi(0) = f_i$

(ii) $\Phi(1) = f_i' \in \text{Emb}\,(S^i_{\lambda-1} \times D^i_{n-\lambda} \to \partial M_n - \text{Image}\, f_j)$

(iii) For $0^i_{n-\lambda} =$ center of $D^i_{n-\lambda}$

and some $x_0 \in S_{n-\lambda-1}$, the isotopy:

$$\Phi \mid S^i_{\lambda-1} \times 0^i_{n-\lambda} : I \to \text{Emb}\,(S^i_{\lambda-1} \times 0^i_{n-\lambda} = S^i_{\lambda-1} \to \partial(M_n + (f_j)))$$

"sweeps exactly once across $D_\lambda{}^j \times x_0$" (see [4], Lemma 2.4). By definition, the handle decomposition:

$$M_n \times (f_1) + \cdots + (f_{i-1}) + (f_i') + (f_{i+1}) + \cdots (f_h) \tag{2}$$

is obtained from (1) by an elementary sliding. ((1) and (2) represent the same manifold.)

By definition, two handle-decompositions

$$M_n + (f_1) + \cdots (f_h) \qquad \text{index}\, f_i = \lambda \qquad (i = 1 \cdots h)$$

$$M_n + (g_1) + \cdots (g_h) \qquad \text{index}\, g_i = \lambda \qquad (i = 1 \cdots h)$$

of the *same* manifold are *equivalent* if they can be obtained one from the other by a finite number of elementary sliding operations.

In general we shall consider a handlebody decomposition:

$$X_n + (f^1_{i_1}) + (f^2_{i_2}) + (f^3_{i_3}) + \cdots (f^h_{i_h})$$

with $i_j = \text{index}\, f^j_{i_j}$. (And throughout the paper, we shall keep from now on the convention that the lower subscript is the index of the handlebody.) The expansion above, is as usually defined by induction, and strictly speaking it should be read

$$(\cdots (((X_n + (f^1_{i_1})) + (f^2_{i_2})) + (f^3_{i_3})) \cdots) + (f^h_{i_h})).$$

A handle decomposition like (1), where all indices are the same, and the images of $S_{\lambda-1} \times D_{n-\lambda}$ two by two disjoint, will be called *proper*. (Then we do not need prentices.) All handlebody-decompositions of the type

$$X_n + (g_2{}^1) + (g_2{}^2) + \cdots (g_2{}^k) \qquad \text{index}\,(g_2{}^i) = 2$$

considered in this paper will be proper. The notion of equivalence, defined before, refers, strictly, to proper handlebody decompositions.

We need, for general handlebody decompositions, a more general notion, *isomorphism*, which will have the property that two proper handlebody decompositions are isomorphic if and only if they are equivalent. (Isomorphism will be also an equivalence relation.)

So, let us consider two general handlebody decompositions, φ and $\bar{\varphi}$: (of the same manifold)

$$\varphi : X_n = Y_n + (f_{i_1}^1) + \cdots + (f_{ik}^k)$$
$$\bar{\varphi} : X_n = Y_n + (\bar{f}_{i_1}^1) + \cdots + (\bar{f}_{i_k}^k).$$

By definition, φ and $\bar{\varphi}$ are *isomorphic* if the following thing happens: $\bar{f}_{i_1}^1$ is obtained from $f_{i_1}^1$ through a global isotopy of ∂Y_n. (This in particular gives a canonical diffeomorphism $\partial(Y_n + (f_{i_1}^1)) = \partial(Y_n + (\bar{f}_{i_1}^1)$.)

$f_{i_2}^2$ is obtained from $f_{i_2}^2$ through a global isotopy of $\partial(Y_n + (f_{i_1}^1)) = \partial(Y_n + (\bar{f}_{i_1}^1))$ (canonical diffeomorphism). This, in particular, gives rise to a canonical diffeomorphism:

$$\partial(Y_n + (f_{i_1}^1) + (f_{i_2}^2)) = \partial(Y_n + (\bar{f}_{i_1}^1) + (\bar{f}_{i_2}^2))).$$

We reserve the name: "*φ is identical to φ'*" for the following stronger notion:

for each j: $f_{i_j}^j = \bar{f}_{i_j}^j$ (maybe up to an isotopy in $S_{i_j-1} \times D_{n-i_j}$).

Definition 1. We denote by $T(n, k)$ the differentiable manifold: (We denote by $\#$ the *connected sum*)

$$T(n, k) = \underbrace{(S_1 \times D_{n-1}) \# \cdots \# (S_1 \times D_{n-1})}_{k \text{ times}} = D_n + (f^1) + \cdots + (f^k)$$

$$(\text{index } f^i = 1 \text{ for } i = 1, \ldots, k) \qquad (T(n, 0) = D_n).$$

Definition 2. A handlebody decomposition:

$$T(n, k) + (g^1) + \cdots + (g^h)$$

with $h \leq k$, index $g^i = 2$ (for $i = 1, \ldots, h$) is called *strictly trivial* if there exists a handle-decomposition:

$$T(n, k) = D_n + (f^1) + \cdots + (f^h) \qquad (\text{index } f^i = 1)$$

such that:

(i) $g^i(S_1{}^i \times o_{n-2}^i) \cap f^j(o_1{}^j \times S_{n-2}^j) = \phi \qquad \text{if } i \neq j$

(ii) $g^i(S_1{}^i \times o_{n-2}^i) \cap f^i(o_1{}^i \times S_{n-2}^i) \qquad (i = 1, \ldots, h)$

contains exactly one point, and the intersection is transversal.

Definition 3. Let us consider a handle of index 2, $\varphi: S_1 \times D_{n-2} \to \partial X_n$ added to the manifold X_n. We shall say that φ is *contractible* if there exists an embedding

$$F: D_2 \times D_{n-3} \to \partial X_n$$

such that, if $S_1 \times I$ is the tubular neighborhood of $S_1 = \partial D_2$ in D_2, then

$$F \mid S_1 \times I \times D_{n-3} = F \mid S_1 \times \underbrace{(I \times D_{n-3})}_{D_{n-2}} = \varphi.$$

When we shall start with a handlebody decomposition

$$Y_n = X_n + (f_2^1) + \cdots + (f_2^k) \qquad (\text{index} f_2^i) = 2$$

and add to it a contractible handle φ:

$$Y_n \# (S_2 \times D_{n-2}) = Y_n + (\varphi) = X_n + (f_2^1) + \cdots + (f_2^k)$$

we shall always assume that

$$\text{Image } F \subset \partial X_n - \cup \text{ Image } (f_2^i).$$

Similarly, when we add several contractible handles (of index 2).

Definition 4. Let us consider the strictly trivial handlebody decomposition:

$$D_n = (S_1 \times D_{n-1}) + (\psi_2^0)$$

where ψ_2^0 is a handle of index 2, obtained as follows: We consider

$$\partial D_n = S_{n-2} = D_n^+ + D_n^- \text{ (northern + southern hemispheres)}$$

and

$$\psi_2^0 = S_1 \times D_{n-2} \to S_1 \times D_{n-2}^+ \qquad \text{(natural diffeomorphism)}$$

We can form the connected sum of the handlebody decomposition

$$X_n = T(n, k) + (g_2^1) + \cdots + (g_2^h)$$

with

$$D_n = (S_1 \times D_{n-1}) + (\psi_2^0) = T(n, 1) + (\psi_2^0)$$

and get a new handlebody decomposition of X_n:

$$X_n = T(n, k + 1) + (g_2^1) + \cdots + (g_2^h) + (\psi_2^0)$$

(compare with section 4 where the operation ρ is defined).

A handlebody decomposition Φ:

$$\Phi\colon X_n = T(n, k) + (g_2^1) + \cdots + (g_2^h)$$

(index $(g_2^i) = 2$, $h \leq k$) is called *trivial* if the following situation occurs: after adding (to Φ) p contractible 2-handles and making its connected sum with q copies of the handlebody decomposition

$$D_n = (S_1 \times D_{n-1}) + (\psi_2^0)$$

we obtain a handlebody decomposition:

$$\Psi : X_n \# \underbrace{(S_2 \times D_{n-2}) \# \cdots \# (S_2 \times D_{n-2})}_{p \text{ times}}$$

$$= T(n, k + q) + (f_2^1) + \cdots + (f_2^{h+p+q})$$

which is equivalent, to a handlebody decomposition Ψ', which can be described as follows:

Ψ' is obtained by adding p contractible handles to a trivial handlebody decomposition:

$$T(n, k + q) + (l_2^1) + \cdots + (l_2^{h+q}).$$

We can state now our main result:

THEOREM A. *Let X_n be a differentiable manifold with a handlebody decomposition*:

$$X_n = T(n, k) + (g^1) + \cdots + (g^l), \qquad n \geq 3$$

where $k, l \geq 0$, index $(g^i) \geq 3$ $(i = 1, \ldots, l)$. Assume also that X_n admits a handle decomposition:

$$X_n = T(n, m) + (f^1) + \cdots (f^p) \tag{3}$$

with $m \geq p \geq 0$, index $(f^i) = 2$ $(i = 1, \ldots, p)$. Then the handle-decomposition (3) *is trivial.*

COROLLARY B. "*Any handlebody decomposition of D_n, of the form*:

$$D_n = T(n, m) + (f^1) + \cdots + (f^m)$$

index $f_i = 2$ $(i = 1, \ldots, m)$ is trivial.

Remark 1. For $n = 2$ the theorem and corollary become trivial.

Problem. Can one replace trivial, by strictly trivial, in the statements above? (at least for $n = 3$?)

The idea of the proof is the following:

We consider *the space $\mathcal{M}(X_n)$ of C^∞ functions $X_n \to R$* (with the C^r topology). This is a connected set, and any two handle-decompositions of X_n induce (roughly speaking) two "generic" elements $f, g \in \mathcal{M}(X_n)$. f and g can be joined by a *generic arc* in $\mathcal{M}(X_n)$; i.e. one can go from one handle-decomposition to another by a "one parameter" family of handle-decompositions of a certain type. Handlebody decompositions are "catastrophically changed" during the process, but nevertheless some things remain invariant, and that enables us to get Theorem A. We remark that our results do *not* require $n \geq 5$.

§2. THE SPACE $\mathcal{M}(X_n)$

The proofs of the statements in this paragraph will be contained in a forthcoming paper by Cerf [1]. Let X_n be a compact differentiable manifold and $\mathcal{M}(X_n)$ the space of differentiable functions $X_n \to R$, endowed with the C^p-topology (p large). Inside $\mathcal{M}(X_n)$ we consider the subset

$$\text{Morse}\,(X_n) \subset \mathcal{M}(X_n)$$

of Morse (i.e. *generic*) functions. We recall that a function f: $X_n \to R$ is a Morse function if it has only nondegenerate critical points, and the critical values separate the critical points.

Morse (X_n) is open and dense in $\mathcal{M}(X_n)$. Let us denote by $F(X_n)$ the closed subset of non-generic functions:

$$F(X_n) = \mathcal{M}(X_n) - \text{Morse}\,(X_n).$$

Let us consider inside $F(X_n)$ the subset $F_n(X_n) \to R$, which fulfill one of the following two requirements:

(Type a) Either f has only nondegenerate critical points but there are exactly two critical points c_1, c_2 such that $f(c_1) = f(c_2)$, or

(Type b) The critical values of f separate the critical points and all the critical points are nondegenerate, except for one and in the neighborhood of that one f can be written (after a convenient coordinate change) in the following form:

$$f = c + \varepsilon_1 x_1^{\,2} + \varepsilon_2 x_2^{\,2} + \cdots + \varepsilon_{n-1} x_{n-1}^2 + x_n^{\,3} \qquad \varepsilon_i = \pm 1.$$

Let us denote by $F_2(X_n) = F(X_n) - F_1(X_n)$.

One can show that $F_1(X_n)$ is open and dense in $F(X_n)$. Moreover "$F_1(X_n)$ is of codimension 1 in $\mathcal{M}(X_n)$" which means: for every $p \in F_1(X_n)$ there exists neighborhoods U of p in $F_1(X_n)$, V of p in $\mathcal{M}(X_n)$ such that:

(i) $V \cap F(X_n) = U$

(ii) There is a homeomorphism

$$V \to U \times R$$

mapping U into $U \times o$.

Also, "$F_2(X_n)$ is of codimension ≤ 2 in $\mathcal{M}(X_n)$" which means: every arc $\alpha\colon I \to \mathcal{M}(X_n)$ such that $\alpha(0), \alpha(1) \notin F_2(X_n)$ can be approximated by an arc $\beta\colon I \to \mathcal{M}(X_n)$ such that $\beta(0) = \alpha(0)$, $\beta(1) = \alpha(1)$, $\beta(I) \cap F_2(X_n) = \phi$.

At this stage it is maybe convenient to talk about a "differentiable" structure for $\mathcal{M}(X_n)$. This can be done either in the language of Banach-manifolds, or, more simply, for our purposes, in the following way:

If Y is a differentiable manifold and $g\colon Y \to \mathcal{M}(X_n)$ a map, g will be called *differentiable* if the induce function $Y \times X_n \to R$ is differentiable. By definition, continuous maps which preserve differentiable maps $Y \to \cdots$ (various differentiable manifolds Y), are differentiable.

A *generic* arc $\alpha\colon I \to \mathcal{M}(X_n)$ is, by definition, a differentiable map $I \to \mathcal{M}(X_n)$ such that:

(i) $\alpha(o), \alpha(1) \in \text{Morse}\,(X_n)$

(ii) $\alpha(I) \cap F_2(X_n) = \phi$

(iii) $\alpha(I) \cap F_1(X_n)$ consists of exactly a finite number of points (the *singular points* of α) and at each of these points, α cuts $F_1(X_n)$ *transversally* (this makes sense because of the fact that codim $F_1(X_n) = 1$; see the precise definition of this, above).

Since $\mathcal{M}(X_n)$ is connected, an obvious application of transversality theory tells us:

LEMMA 1. "*If* $\alpha(o)$, $\alpha(1) \in$ *Morse*(X_n), *there exists a generic arc* $\alpha\colon I \to \mathcal{M}(X_n)$ *joining* $\alpha(o)$ *and* $\alpha(1)$.

We remark that if $\alpha(I) \cap F(X_n) = \phi$, then $\alpha(o)$ and $\alpha(1)$ are the same, up to a diffeomorphism of X_n.

We want to describe generic arcs, and for this purpose we recall the connection in between Morse functions and handle-adding (see [3]). *From now on, we consider only functions* $X_n \to R$ *which are constant on each arcwise connected component of* ∂X_n and for them, everything we said before, still applies. (For example, they still form an arcwise connected set because, if $f, g \in \mathcal{M}(X_n)$ are constant on each arcwise connected component of ∂X_n, so is $tf + (1-t)g$ for each $o \leq t \leq 1$.)

If $f\colon X_n \to R$ is a Morse function, we can attach to it a handle decomposition of X_n:

$$\Phi(f)\colon X_n = \phi + (\varphi^1_{i_1}) + (\varphi^2_{i_2}) + \cdots + (\varphi^k_{i_k})$$

where $i_j = \text{index}\ (\varphi^j_{i_j}$ and where $i_1, i_2, \ldots, i_k$ is exactly the sequence of the indexes of the critical points of f, ordered after their increasing values.

Conversely, if

$$\psi\colon X_n = \phi + (\psi^1_{i_1}) + \cdots + (\psi^l_{i_l}) \tag{4}$$

(index $(\psi^j_{i_j}) = i_j$) is a handle decomposition of X_n, we can attach to it a Morse function $\Psi(\psi)\colon X_n \to R$ such that $i_1, \ldots, i_l$ is exactly the sequence of the indexes of the critical points of $\Psi(\psi)$, ordered after their increasing values. $\Psi(\psi)$ is uniquely determined up to a diffeomorphism of X_n.

Conversely $\Phi(f)$ is uniquely determined up to "isomorphism." Up to isomorphism, of diffeomorphism, we have:

$$\Psi\Phi(f) = f, \qquad \Phi\Psi(\varphi) = \varphi.$$

We can describe now the generic arc, "locally" (locally in the parameter space). As long as $\alpha(t)$ does not meet $F(X_n)$, $\Phi(\alpha(t))$ does not change, up to isomorphism.

If $\alpha(t_0) \in F_1(X_n)$ is of the type (a) then one passes from $\Phi(\alpha(t_0 - \varepsilon))$ to $\Phi(\alpha(t_0 + \varepsilon))$ as follows:

$\Phi(\alpha(t_0 - \varepsilon))$ (up to isomorphism) is:

$$X_n = \phi + (\varphi^1_{i_1}) + (\varphi^2_{i_2}) + \cdots + (\varphi^l_{i_l}) + (\varphi^{l+1}_{i_{l+1}}) + \cdots + (\varphi^k_{i_k})$$

and Image $(\varphi_i{}^l) \cap$ Image $(\varphi^{l+1}_{i_{l+1}}) = \phi$ (inside $\partial(\phi + (\varphi^1_{i_1}) + \cdots + (\varphi^{l-1}_{i_{l-1}}))$.

$\Phi(\alpha(t_0 + \varepsilon))$ is:

$$X_n = \phi + (\varphi^1_{i_1}) + \cdots + (\varphi^{l-1}_{i_{l-1}}) + (\varphi^{l+1}_{i_{l+1}}) + (\varphi^l_{i_l}) + (\varphi^{l+2}_{i_{l+2}}) + \cdots + (\varphi^k_{i_k}).$$

("interchanging two handles, i.e. two critical values.")

If $\alpha(t_0) \in F(X_n)$ is of the type (b) then arc passes from $\Phi(\alpha(t_0 - \varepsilon))$ to $\Phi(\alpha(t_0 + \varepsilon))$ either by *cancelling* two successive handles of index i, $i + 1$ (see Lemma 2.2 in [4]), or by inserting in between two consecutive handles a new pair of handles, which are cancelling each other.

A convenient geometrical picture for $\alpha(t)$ is a *Cerf diagram*, which is obtained as follows: for each $\alpha(t)$ we consider in the (t, x)-plane the set $(t, c_0), (t, c_1) \ldots$ where c_i are the various critical values of $(\alpha(t))$. By varying $t \in I$ we get a set $\Delta(\alpha) \subset (t, x)$-plane, which we call a Cerf diagram. $\Delta(\alpha)$ is essentially of the graph of a family of curves with a well-defined projection $\pi\colon \Delta(\alpha) \to t$-axis $(\pi(t, c) = t)$. If $\alpha(t_0 - \varepsilon, t_0 + \varepsilon) \cap F(X_n) = \phi$, then $\pi^{-1}(t_0 - \varepsilon, t_0 + \varepsilon)$ is a disjoint family of graphs of curves $c_i = c_i(t)$. Corresponding to points in $\alpha(t) \cap F(X_n)$ of type (a) or (b), $\Delta(\alpha)$ has *crossing points* or *cusps*. (Birth or death points for pairs of mutually cancelling handles.)

§3. THE MAIN CONSTRUCTION

We shall start with some handlebody decomposition:

$$\varphi : X = \phi + (\varphi^1_{i_1}) + (\varphi^2_{i_2}) + \cdots + (\varphi^k_{i_k}) \qquad (X_n \text{ connected}).$$

We shall attach to φ another (proper) handelbody decomposition:

$$\Gamma(\varphi) : Y_n = T(n, l) + (\psi^1) + \cdots + (\psi^m) \qquad (\text{index } \psi^i = 1)$$

determined up to equivalence (see Section 1).

For this purpose, we recall the following standard fact from handlebody theory:

Assume one has a handlebody decomposition:

$$M_n = N_n + (\varphi_i^1) + (\varphi_j^2) \qquad j \le i$$

then up to isomorphism (more precisely up to isotopy in $\partial(N_n + \varphi_i^1)$) one can assume that

$$\text{Image } (\varphi_j^2(S^2_{j-1} \times D^2_{n-j})) \subset \partial(N_n + (\varphi_i^1)) \cap \partial N_n. \tag{5}$$

We recall that this is performed by sliding $\varphi_j^2(S^2_{j-1} \times o^2_{n-j})$ across $D_i^1 \times S^1_{n-1} \subset \partial(N_n + (\varphi_i^1))$, which is possible since $j - 1 < i$.

In usual terms, this means that φ_j^2 can be isotoped (inside $\partial(N_n + (\varphi_i^1))$) to a $\bar{\varphi}_j^2$ which has property (5) and

$$N_n + (\varphi_i^1) + (\varphi_j^2) \underset{\text{isomorphism}}{=} N_n + (\varphi_2^1) + (\bar{\varphi}_j^2) \underset{\text{diffeomorphism}}{=} N_n + (\bar{\varphi}_j^2) + (\varphi_i^1).$$

Our problem is how much is $\bar{\varphi}_j^2$ determined (up to isotopy in ∂N_n).

LEMMA 2. *If $j = 2$, $i > 2$, then the isotopy class of $\bar{\varphi}_j^2$ in ∂N_n is uniquely determined.*

Proof. For $n = 3$ the lemma is trivial. For $n \ge 4$ the lemma can be easily reduced to the proof of the following local situation:

PROPOSITION. *Let us consider $R_n = R_i \times R_{n-i}$, $n \ge 4$, $i \ge 3$ and in the R_i-space the two concentric disks $d_i \subset D_i$ of boundaries s_{i-1} and S_{i-1}. Let us consider two embeddings:*

$$f, g : I \times D_{n-1} \to R_n - (d_i \times R_{n-1}) \text{ such that:}$$

(i) $f \mid x \times D_{n-1} = g \mid x \times D_{n-1}$ for $x = o$ or 1

(ii) $f(I \times o), g(I \times o) \subset S_{i-1} \times o$ $\quad o \in R_{n-i}$

(iii) f and g are isotopic (keeping $f \mid \partial I \times D_{n-1} = g \mid \partial I \times D_{n-1}$ fixed) in R_n.

Then f, g are isotopic (keeping the boundary fixed, again) in $R_n - (d_i \times R_{n-i})$.

The proof is immediate. $f \mid I \times o$ and $g \mid I \times o$ are clearly isotopic, keeping the boundary fixed (since they are homotopic, being contained in S_{i-1}, $i - 1 \ge 2$, and $n \ge 4$). So the only problem is about the normal handle, which is determined by an element in $\pi_1(0_{n-1}) = Z_2$, independently of the fact that we consider R_n on $R_n - (d_i \times R_{n-i})$ (see [2]).

We have also the trivial lemma:

LEMMA 3. *If $j = 1$, $i > 1$ then the isotopy class of $\bar{\varphi}_j^2$ in ∂N_n is uniquely determined.*

Remark. Statements like Lemmas 2, 3 are *not* true, in general, for all j's. This motivates the choice of dimensions in our Theorem A.

Now $\Gamma(\varphi)$ is defined as follows: through the handle-permutation procedure which we have recalled before, in the handle-decomposition (4) $(\varphi : X_n = \phi + (\varphi_{i_1}^1) + \cdots)$ we can push the handles of index o in front of all the others, afterwards the handles of index 1, and afterwards the handles of index 2. We get a new handle-decomposition:

$$\psi : X = \phi + \underbrace{(\varphi_0{}^1) + \cdots + (\varphi_0{}^\alpha)}_{\text{handles of index 0}} + \underbrace{(\varphi_1{}^1) + \cdots + (\varphi_1{}^\beta)}_{\text{index 1}}$$
$$+ \underbrace{(\psi^1) + \cdots + (\psi^m)}_{\text{index 2}} + (\text{Handles of index} > 2). \quad (6)$$

Since X_n is assumed connected, $\beta - \alpha = l \geq 0$, and

$$\phi + (\varphi_0{}^1) + \cdots + (\varphi_1{}^1) + \cdots + (\varphi_1{}^\beta) = T(n, l).$$

By Lemma 2, independently of the way in which we have permuted the handles of φ, the handle-decomposition:

$$Y_n = T(n, l) + (\psi^1) + \cdots + (\psi^m)$$

is uniquely defined, up to isomorphism. By definition, this is $\Gamma(\varphi)$.

Clearly if ψ is the handle decomposition:

$$\psi : X_n = T(n, l) + (\psi_2{}^1) + \cdots + (\psi_2{}^m) + (\text{handles of index} > 2)$$

then $\Gamma(\psi)$ is the handle-decomposition

$$T(n, l) + (\psi_2{}^1) + \cdots + (\psi_2{}^m).$$

Or, alternatively $\Gamma\Gamma(\psi) = \Gamma(\psi)$.

We want to study now the effect of a generic path in the space of Morse functions on the operation Γ. But before that we need some more preparation.

§4. $*$-HANDLE-DECOMPOSITIONS

To simplify the language we shall call from now on a (proper) handle-decomposition of the form:

$$T(n, l) + (\psi^1) + \cdots + (\psi^k) \qquad \text{index } (\psi^i) = 2$$

a *$*$-handle-decomposition*. ($\Gamma(\varphi)$ is a $*$-handle-decomposition.) In section 1 we have defined an equivalence relation for $*$-handle-decompositions.

Let ϕ be a $*$-handle-decomposition:

$$\varphi : Y = T(n, l) + (\psi^1) + \cdots + (\psi^k).$$

We consider also the $*$-handle-decomposition

$$\psi : D_n = S_1 \times D_n + (\psi) \qquad \text{(see Section 1)}$$

where $\partial D_n = S_{n-1} = D_{n-1}^+ + D_{n-1}^- =$ northern hemisphere + southern hemisphere, and ψ is the handle of index 1 given by:

$$\psi : S_1 \times D_{n-1} \underset{\text{natural diffeomorphism}}{\rightarrow} S_1 \times D_{n-1}^+$$

We define a new $*$-handle-decomposition

$$\rho(\varphi): Y_n = (T(n, l) \# (S_1 \times D_n)) + (\psi^1) + \cdots + (\psi^k) + (\psi). \tag{7}$$

It is understood here that $\#$ is the connected sum of the two manifolds with boundary $T(n, l)$ and $(S_1 \times D_n)$ and that this connected sum is defined by two $(n-1)$-disks in $\partial T(n, l)$ and $\partial(S_1 \times D_n)$, not touching Image (ψ^i), Image (ψ), respectively.

If $n \geq 4$ then $\rho(\varphi)$ is uniquely determined, ("up to identity") as a $*$-handlebody-decomposition (we observe that:

$$T(n, l) \# (S_1 \times D_n) = T(n, l+1))$$

but if $n = 3$ this is not necessarily so, since $\partial T(3, l) - \cup$ Image (ψ^i) is not necessarily connected. But one checks that *the equivalence class of* $\rho(\varphi)$ *is uniquely determined* (one checks this by sliding the ψ^i's over ψ in formula (7)).

So if $n \geq 4$, $\rho(\varphi)$ will be defined as a $*$-handlebody decomposition, but if $n = 3$ only as an equivalence class of $*$-handlebody decompositions.

It is clear, anyway, that if φ and φ' are equivalent $*$-handlebody decompositions, then $\rho(\varphi)$ and $\rho(\varphi')$ are equivalent (since the $\#$ does not affect the slidings of $\varphi \to \varphi'$).

We give the following lemma which will not be used afterwards in this paper (but is a step in replacing "trivial" by "strictly trivial" in our main Theorem and corollary).

LEMMA 4. *Let us consider a $*$-handlebody decomposition*:

$$\varphi: Y_n = T(n, k) + (\psi^1) + \cdots + (\psi^h) \qquad h \leq k$$

and the $*$-handlebody decomposition $\rho(\varphi)$, *which we denote (for convenience) by*:

$$\rho(\varphi): Y_n = T(n, k+1) + (\psi^0) + \cdots + (\psi^k).$$

Then φ *is strictly trivial if and only if* $\rho(\varphi)$ *is strictly trivial.*

Remark. In our present situation the ambiguity in the definition of $\rho(\varphi)$ for $n = 1$ vanishes. So this lemma refers to $*$-handlebody decompositions, not just to equivalence classes.

Proof. If φ is strictly trivial, then $\rho(\varphi)$ is strictly trivial.

Let us assume that $\rho(\varphi)$ is strictly trivial. That means, precisely, the following thing:

In $T(n, k+1)$, there exists $h + 1$ disks of dimension n: $D_n^0, \ldots, D_n^h$ such that:

(a) $D_n^i \cap \partial T(n, k+1) = \partial D_n^i$ and the intersection is transversal.

(b) By "cutting" $T(n, k+1)$ along the D_n^i's, we get exactly $T(n, k-h)$ (and more generally by cutting $T(n, k+1)$ along any subset of $(s+1)$-elements of the D_n^α's a $T(n, k-s)$).

(c) $S_{n-1}^i \cap \psi^j(S_1 \times o) = \phi$ if $i \neq j$ and exactly a point if $i = j$. The intersection is transversal. $(S_{n-1}^i = \partial D_n^i)$.

We would like to prove a similar fact for φ. Let us denote by Θ_o the result of cutting $T(n, k+1)$ along D_n^0. We remark that the sets $\psi^i(S_1{}^i \times D_{n-1}^i)$ are naturally embedded in

$\partial\Theta_0$. If we can prove that there exists a diffeomorphism of pairs:

$$(\Theta_0, \cup \text{ Image } \psi^i) \xrightarrow{g} (T(n, k+1), \cup \text{ Image } \psi^i),$$

which induces the identity on $\cup$ Image ψ^i everything would be finished. We can realize g as the composition of the following two diffeomorphisms, which induce the identity on $\cup$ Image ψ^i.

I. The diffeomorphism:

$$T(n, k) = T(n, k) \# D_n = T(n, k) \# ((S_1 \times D_n) + (\psi^0)) = T(n, k+1) \# (\psi^0)$$

II. The diffeomorphism:

$$\Theta_0 = \Theta_0 \# D_n = \Theta_0 \# (D_n{}^0 + nbd\ D^i_{n-1}) = T(n, k+1) \# (\psi^0)$$

(we denote here by $nbd\ D^i_{n-1}$ a tubular neighborhood of D^i_{n-1} in $T(n, k+1)$).

This finishes the proof of Lemma 4.

Problem. If $\rho(\varphi)$ is equivalent to a strictly trivial handlebody decomposition, is then φ strictly trivial (or at least equivalent to a strictly trivial handlebody decomposition)?

LEMMA 5. *Let us consider the same conditions as in lemma* 4. *Then φ is trivial if and only if $\rho(\varphi)$ is trivial.*

Remark. This is now a lemma about equivalence classes. It is not a consequence of Lemma 4, but a much easier fact.

Proof. If φ is trivial, that means that after adding to it q contractible handles $(f^1), \ldots, (f^q)$ and applying the operation ρ, p times, we get a handlebody decomposition φ' which is equivalent to a handle-decomposition φ'' obtained by adding q contractible handles to a trivial handlebody decomposition. But φ' can be obtained from $\rho(\varphi)$ by adding the contractible handles $(f^1), \ldots, (f^q)$ and applying ρ $(p-1)$ times. So $\rho(\varphi)$ is trivial.

One can reverse the argument and prove that if $\rho(\varphi)$ is trivial, so is φ.

We also have:

LEMMA 6. *Let φ be a $*$-handle-decomposition and let us denote by $\gamma(\varphi)$ the $*$-handle-decomposition obtained from it by adding one contractible handle to φ.*

φ is trivial if and only if $\gamma(\varphi)$ is trivial.

The proof is deduced directly from the definition, like the proof of Lemma 5.

§5. THE PROOF OF THEOREM A

We need three lemmas. These lemmas will refer to a generic path $\alpha: I \to (X_n)$.

LEMMA 7. *If $\alpha(I)$ has no singular points, then $\Gamma(\Phi(\alpha(0)))$ and $\Gamma(\Phi(\alpha(1)))$ are equivalent.*

LEMMA 8. *If $\alpha(I)$ has exactly one singular point $t_o \in \text{int } I$, such that $\alpha(t_0) \in F_1(X_n)$ is of type* (*a*) (*see Section* 2), *then $\Gamma(\Phi(\alpha(0)))$ and $\Gamma(\Phi(\alpha(1)))$ are equivalent.*

LEMMA 9. *If $\alpha(I)$ has exactly one singular point $t_o \in I$, such that $\alpha(t_0) \in I_1(X_n)$ is of type* (*b*), *then one of the following five situations occurs:*

(9.1) $\Gamma(\Phi(\alpha(o))) = \Gamma(\Phi(\alpha(1)))$ ("$=$" means equivalence)

(9.2) $\gamma(\Gamma(\Phi(\alpha(o)))) = \Gamma(\Phi(\alpha(1)))$

(9.3) $\Gamma(\Phi(\alpha(0))) = \gamma(\Gamma(\Phi(\alpha(1))))$

(*see notation of Lemma* 6)

(9.4) $\rho(\Gamma(\Phi(\alpha(0)))) = \Gamma(\Phi(\alpha(1)))$

(9.5) $\Gamma(\Phi(\alpha(0))) = \rho(\Gamma(\Phi(\alpha(1))))$.

Before we give the proof of these lemmas, let us show how they imply Theorem A (Corollary B is afterwards an immediate consequence).

Let us consider the two handlebody decompositions of X_n, from the statement of Theorem A:

$$\varphi: X_n = T(n, k) + (g^1) + \cdots + (g^l) \qquad (\text{index}\,(g^i) \geq 3)$$

$$\varphi': X_n = T(n, m) + (f^1) + \cdots + (f^p) \qquad 0 \leq p \leq m,$$

index $f^i = 2$.

We consider the $*$-handlebody decomposition: $\Gamma(\varphi) = T(n, k)$ which is (strictly) trivial and $\Gamma(\varphi') = \varphi'$. We consider the Morse functions $\Psi(\varphi)$ and $\Psi(\varphi')$ (together with the handlebody decompositions attached to them:

$$\Phi(\Psi(\varphi)) = \varphi, \qquad \Phi(\Psi(\varphi')) = \varphi').$$

$\Psi(\varphi)$ and $\Psi(\varphi')$ can be joined in $\mathcal{M}(X_n)$ by a generic arc $\alpha: I \to \mathcal{M}(X_n)$ with $\alpha(0) = \Psi(\varphi)$, $\alpha(1) = \Psi(\varphi')$. We consider the various $\Gamma(\Phi(\alpha(t))) = \Gamma(t)$ for $t \in I$. $\Gamma(t)$ is a family of $*$-handle-decompositions which is constant as long as t does not pass through a singular value, where $\Gamma(t)$ gets changed by equivalence or one of the operations ρ, γ. $\Gamma(0) = T(n, k)$ is trivial hence so is $\Gamma(1) = \varphi'$. This completes the proof of Theorem A, provided that Lemmas 7, 8 and 9 are proved.

Proof of Lemmas 7 *and* 8. Lemmas 7 and 8 amount to prove that if φ is a handle-decomposition, then the equivalence class of the $*$-handle-decomposition $\Gamma(\varphi)$ is completely determined by the equivalence class of φ. This can be reduced to Lemma 2 and the following fact: suppose we have a handle-decomposition

$$\Psi: N_n + (\psi_i) + (\psi_2) \qquad \text{where index}\,(\psi_i) = i \geq 2.$$

By sliding (ψ_2) over (ψ_i) we get an embedding.

$\bar{\psi}_2: S_1 \times D_{n-2} \to \partial N_n$ (whose isotopy class, by Lemma 2, is unique). Suppose we change ψ up to equivalence, first, i.e. isotop ψ_i and then isotop ψ_2. After that slide ψ_2 over ψ_i to get an embedding $\bar{\psi}_2: S_1 \times D_{n-2} \to \partial N_n$. The embeddings $\bar{\psi}_2$ and $\bar{\psi}_2$ are isotopic (and this is easily checked). The case when one permutes a handle of index 1 with a handle of index 2 is trivial.

Proof of Lemma 9. Cases 9.2 (and 9.3) correspond to the creation (annihilation) of a pair of mutually cancelling handles of index 2 and 3. Cases 9.4 (and 9.5) correspond to the creation (annihilation) of a pair of mutually cancelling handles of index 1 and 2.

The rest of creations and/or annihilations of pairs of mutually cancelling handles, correspond to the case 9.1.

To give the proof, we have to describe more explicitly what a creation (and/or an annihilation) of a pair of mutually cancelling handles means.

Suppose we have a handlebody decomposition

$$\varphi : X_n = T(n, m) + (\varphi^1_{i_1}) + (\varphi^2_{i_2}) + \cdots + (\varphi^k_{i_h}).$$

(For simplicity we shall omit the indexes i_j, from now on.) Let Y_n be:

$$Y_n = T(n, m) + (\varphi^1) + \cdots + (\varphi^l) \tag{8}$$

We consider a succession of 2 handles added to (8)

$$T(n, m) + (\varphi^1) + \cdots + (\varphi^l) + (\psi_i) + (\psi_{i+1}) \tag{9}$$

where we assume that

$$o_i \times S_{n-i-1} \subset \partial(D_i \times D_{n-i}) \cap \partial(T(n, m) + (\varphi^1) + \cdots + (\psi_i))$$

and $\psi_{i+1}(S_i \times o_{n-i-1})$ have exactly one point in common and the intersection is transversal.

It is well-known (see [4]) that (9) and (8) represent diffeomorphic manifolds, so it makes sense to consider the new handlebody decomposition of X_n (which as far as $(\varphi^{l+1}), \ldots,$ are concerned, is defined only up to isomorphism):

$$\varphi' : X_n = T(n, m) + (\varphi^1) + \cdots + (\varphi^l) + (\psi_i) + (\psi_{i+1}) + (\varphi^{l+1}) + \cdots + (\varphi^k). \tag{10}$$

That is the creation of a pair of mutually cancelling handles of index i, $i + 1$ (and conversely passing from (10) to (8) is an annihilation).

Now up to isomorphism (more precisely up to an isotopy of ψ_i, ψ_{i+1}), φ' can be described much nicer. We remark that the definition of a *contractible* handle, which was given in Section 1, for handles of index 2, makes sense, in general, for handles of index i. Up to isotopy, we can always assume that ψ_i is added directly to $\partial T(n, m)$, that it is contractible, which means in particular that $\psi_i(S_{i-1} \times o_{n-1})$ bounds an i-disk $\Delta_i \subset \partial T(n, m)$. Moreover, up to izotopy we may assume that

$$\psi_{i+1}(S_i \times o_{n-i-1}) = \Delta_i + D_i \times \omega_{n-i} \qquad \omega_{n-i} \in S_{n-i-1}$$

(this last formula has to be slightly modified to become completely correct, but its meaning is nevertheless clear).

With this new interpretation of ψ_i, ψ_{i+1}, after isotopy (i.e. equivalence) the proof of our lemma is trivial.

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Faculté des Sciences
d'Orsay
Dept. de Math. 91-*Orsay France*